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QUANTIFIER ELIMINATION FOR HENSELIAN FIELDS RELATIVE TO ADDITIVE AND MULTIPLICATIVE CONGRUENCES*

BY

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ABSTRACT

Several classes of henselian valued fields admit quantifier elimination relative to structures which reflect the additive and multiplicative congruences of the field. Value groups and residue fields may be viewed as reducts of these structures. A general theorem is given using the theory of tame extensions of henselian fields. Special cases like the case of p-adically closed fields and the case of henselian fields of residue characteristic 0 are discussed.

Introduction

In this paper, we will consider the following problem:

Given two valued fields $\mathbf{L} = (L, v)$ *and* $\mathbf{F} = (F, v)$ *with a common subfield* $\mathbf{K} = (K, v)$, *find a criterion for* $\mathbf{L}$ *and* $\mathbf{F}$ *to be elementarily equivalent over* $\mathbf{K}$.

Here, we use the following terminology: given a first order language $\mathcal{L}$ and $\mathcal{L}$–structures $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2$ where $\mathcal{S}$ is a common substructure of $\mathcal{S}_1$ and $\mathcal{S}_2$ with universe S, then we will say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are elementarily equivalent over

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$\mathcal{S}$ if they are elementarily equivalent as $\mathcal{L}(S)$–structures. It will be denoted by $\mathcal{S}_1 \equiv_{\mathcal{S}} \mathcal{S}_2$.

The problem as stated above plays a key role for the question whether a given elementary theory $\mathcal{T}$ of valued fields admits quantifier elimination. It is well known that quantifier elimination is equivalent to substructure completeness which means that every two models of $\mathcal{T}$ are equivalent over every common substructure. This is a stronger condition than model completeness which requires only that every two models are elementary equivalent over every common submodel. However, model completeness for elementary theories of valued fields can only be obtained if the corresponding theories of value groups and residue fields are model complete. The most prominent examples for such theories are

– the theory of algebraically closed valued fields (cf. [ROB])

– the theory of (formally) $\wp$–adically closed fields of fixed p–rank (cf. [P–R]).

At the same time, the second example has shown that quantifier elimination for the theory of value groups and for the theory of residue fields is not sufficient to guarantee quantifier elimination for such theories of $\wp$–adically closed fields. The same obstruction is found for other theories of valued fields too, with the exception of algebraically closed valued fields. Some more structure is needed, and in the case of the p–adics it was found by Macintyre [M]. Using Macintyre's n–th power predicates P_n for every $n \in \mathbb{N}$ (where $P_n(x) \Leftrightarrow \exists y$: $y^n = x$), the comprehensive theorem for $\wp$–adically closed fields was given by Prestel and Roquette ([P-R], Theorem 5.6):

THEOREM 1.1: *In the language of valued fields enriched by the predicates P_n, $n \in \mathbb{N}$, the theory of $\wp$–adically closed fields of fixed p–rank d for which P_n is interpreted by the power set K^n, admits elimination of quantifiers.*

Returning to our problem as stated at the beginning, we have seen that the criterion that the value groups as well as the residue fields of **L** and **F** are elementarily equivalent over those of **K**, is not sufficient. So we are led to the question whether there exists a similar structure which will do the job. Such a structure was given by Basarab [B]. He introduced what he called "mixed structures" which connect value group and residue field by groups which are obtained from $K^\times$ by multiplicative congruences. In [B], Basarab has obtained quantifier elimination for henselian fields of characteristic 0 with residue characteristic $p > 0$ relative to these mixed structures.

In the present paper, we will generalize Basarab's result and at the same time simplify the concept of Basarab's mixed structures. The structures that we will consider are structures of additive and multiplicative congruences together with a relation between both. These will be called **amc–structures**. Basarab's result will be put into a larger framework by using the notion of tame extensions. This enables us to give answers to our general problem by relating it to the known model completeness results for elementary theories of valued fields.

For several theories of henselian fields, we will show quantifier elimination (more precisely: substructure completeness) relative to the amc–structures. This yields quantifier elimination in the language of valued fields with respect to suitable classes of predicates which represent formulas of the amc–structures. In this paper, we will not explicitly determine such predicates except for the cases in which the power predicates suffice. Delon [D] has shown quantifier elimination with respect to a generalization of the power predicates for various elementary classes of henselian fields of equal characteristic. Her results were generalized by L. van den Dries [VDD2] to the case of henselian fields of characteristic 0 with residue characteristic $p > 0$. See also Weispfenning's primitive recursive quantifier elimination in [W]. However, all these results do not give a simple answer to our original problem of determining a clear cut structure strengthening the structure of value group and residue field and rendering relative quantifier elimination. But in turn, the proofs given in this paper may be tailored to prove Delon's and van den Dries' results. A systematic treatment of this aspect will be given in [K2].

2. Basic notions and the main theorems

Given a valued field $\mathbf{K} = (K, v)$ let us denote by $\mathcal{O}_{\mathbf{K}}$ the valuation ring, by $\overline{K}$ or Kv the residue field and by vK the value group $vK = \{va \mid 0 \neq a \in K\}$. If there is danger of confusion, we will write $v_{\mathbf{K}}$ to denote to which field the valuation refers.

A subset δ of vK will be called **initial segment** if it satisfies

$$\alpha \in \delta \wedge \beta \in vK \wedge \alpha > \beta \Rightarrow \beta \in \delta \;.$$

The initial segments of vK are ordered by inclusion. Every element of $\alpha \in vK$ may be identified with the smallest initial segment in which it is contained, that is $\{\beta \in vK \mid \alpha \geq \beta\}$; in this way, vK is an ordered subset of the ordered set of all its initial segments. We may thus use δ for initial segments as well as for elements of vK (but if we say $\delta \in vK$, we will always mean an element). Similarly, the convex subgroups $\Delta \leq vK$ will be identified with the smallest initial segment that they are contained in.

For every initial segment δ of vK, let $\mathcal{M}_{\mathbf{K}}^{\delta}$ be the ideal $\{a \in \mathcal{M}_{\mathbf{K}} | \ va > \delta\}$ of $\mathcal{O}_{\mathbf{K}}$. In particular, $\mathcal{M}_{\mathbf{K}}^{0} = \mathcal{M}_{\mathbf{K}}$ is the maximal ideal of the valuation ring $\mathcal{O}_{\mathbf{K}}$. Note that $\mathcal{M}_{\mathbf{K}}^{\delta} = \mathcal{M}_{\mathbf{K}}^{0}$ for every $\delta \leq 0$. Further, $\mathcal{O}_{\mathbf{K}}^{\delta}$ will denote the factor ring $\mathcal{O}_{\mathbf{K}}/\mathcal{M}_{\mathbf{K}}^{\delta}$; this is a local ring with maximal ideal $\mathcal{M}_{\mathbf{K}}/\mathcal{M}_{\mathbf{K}}^{\delta}$. In particular, $\mathcal{O}_{\mathbf{K}}^{0} = \overline{K}$. We write π_δ for the canonical projection $\mathcal{O}_{\mathbf{K}} \longrightarrow \mathcal{O}_{\mathbf{K}}^{\delta}$. Note that for $a \in \mathcal{O}_{\mathbf{K}}$, the projection $\pi_\delta a$ is an invertible element of $\mathcal{O}_{\mathbf{K}}^{\delta}$ if and only if $va = 0$.

On the other hand, consider the multiplicative groups $G_{\mathbf{K}}^{\delta} = K^{\times}/1 + \mathcal{M}_{\mathbf{K}}^{\delta}$. In particular,

$$G_{\mathbf{K}} := G_{\mathbf{K}}^{0} = K^{\times}/1 + \mathcal{M}_{\mathbf{K}} \,.$$

We write π_δ^* for the canonical projection $K^{\times} \longrightarrow G_{\mathbf{K}}^{\delta}$. Note that $G_{\mathbf{K}}^{\delta}$ is the group of multiplicative congruence classes modulo $\mathcal{M}_{\mathbf{K}}^{\delta}$ in the sense of Hasse. The group $G_{\mathbf{K}}$ reminds of the power predicates P_n. Indeed, if $\mathbf{K}$ is henselian and n is not divisible by the residue characteristic of $\mathbf{K}$ then Hensel's Lemma shows that $a \in \mathbf{K}$ admits an n-th root in $\mathbf{K}$ if and only if $\pi_0^* a$ admits an n-th root in $G_{\mathbf{K}}$. If n is divisible by the residue characteristic then this does not work. But if in this case, the characteristic of $\mathbf{K}$ itself is 0, then the groups $G_{\mathbf{K}}^{\delta}$ for $\delta > 0$ may be used to overcome this difficulty, as we will see below.

The local ring $\mathcal{O}_{\mathbf{K}}^{\delta}$ and the group $G_{\mathbf{K}}^{\delta}$ are related through a relation given by

$$\forall x \in \mathcal{O}_{\mathbf{K}}^{\delta}\, \forall y \in G_{\mathbf{K}}^{\delta}: \quad \Theta_\delta(x,y) \Leftrightarrow \exists z \in \mathcal{O}_{\mathbf{K}}\colon \pi_\delta z = x \wedge \pi_\delta^* z = y \,.$$

For elements of value 0, additive congruence modulo $\mathcal{M}_{\mathbf{K}}^{\delta}$ implies multiplicative congruence modulo $1 + \mathcal{M}_{\mathbf{K}}^{\delta}$. Hence Θ_δ induces a group homomorphism from ${\mathcal{O}_{\mathbf{K}}^{\delta}}^{\times}$ into $G_{\mathbf{K}}^{\delta}$ given by

$$\vartheta_\delta:\ a + \mathcal{M}_{\mathbf{K}}^{\delta} \mapsto a(1 + \mathcal{M}_{\mathbf{K}}^{\delta}) \ \text{ for all } a \in \mathcal{O}_{\mathbf{K}}^{\times} \,.$$

We have

(1) $$\pi_\delta^* a = \vartheta_\delta \pi_\delta a \ \text{ for all } a \in \mathcal{O}_{\mathbf{K}}^{\times} \,.$$

For every initial segment δ of vK, we consider the system

$$\mathbf{K}_{\cdot\delta} = (\mathcal{O}_{\mathbf{K}}^{\delta},\ G_{\mathbf{K}}^{\delta},\ \Theta_\delta)$$

and call it the **structure of additive and multiplicative congruences of level δ in K**, or shorter: the **amc–structure of level δ**. In particular, $\mathbf{K}_0$ is the pair $(\overline{K},\ G_{\mathbf{K}})$ together with the embedding

$$\vartheta_0 : \overline{K}^{\times} \longrightarrow G_{\mathbf{K}}$$

whose cokernel is just the value group of $\mathbf{K}$:

$$vK \cong G_{\mathbf{K}}/\vartheta_0\overline{K}^{\times}\ , \tag{2}$$

and together with a unary predicate

$$\mathrm{Pos}(x) :\equiv \Theta_0(0,x)$$

on $G_{\mathbf{K}}$ whose range is exactly $\pi_0^*\mathcal{M}_{\mathbf{K}}$ and which maps modulo $\vartheta_0\overline{K}^{\times}$ onto the subset of positive elements in vK. More generally, on $G_{\mathbf{K}}^{\delta}$ we define

$$\mathrm{Pos}_\delta(x) :\equiv \Theta_\delta(0,x)$$

whose range is exactly $\pi_\delta^*\mathcal{M}_{\mathbf{K}}^{\delta}$. For an arbitrary valued field $\mathbf{K}$,

$$vK \cong G_{\mathbf{K}}/\{g \in G_{\mathbf{K}} \mid \neg\mathrm{Pos}(g) \wedge \neg\mathrm{Pos}(g^{-1})\} \tag{3}$$

and the order on vK (more precisely, the subset of all elements > 0) is just the image of the predicate Pos.

Every formula φ in the language of amc–structures can be encoded by a formula φ_δ in the language of valued fields augmented by a constant symbol for an arbitrary element in K of value δ, such that for all $a_1,\ldots,a_n \in \mathcal{O}_{\mathbf{K}}$ and $b_1,\ldots,b_m \in K$, we have $\mathbf{K} \models \varphi_\delta(a_1,\ldots,a_n,b_1,\ldots,b_m)$ if and only if $\mathbf{K}_\delta \models \varphi(\pi_\delta a_1,\ldots,\pi_\delta a_n,\pi_\delta^* b_1,\ldots,\pi_\delta^* b_m)$.

Let two extensions $\mathbf{L} = (L,v)$ and $\mathbf{F} = (F,v)$ of $\mathbf{K}$ be given. If $\mathbf{L}$ and $\mathbf{F}$ are equivalent over $\mathbf{K}$, then by virtue of our preceding remark, $\mathbf{L}_\delta$ and $\mathbf{F}_\delta$ are equivalent over $\mathbf{K}_\delta$ for each $\delta \in vK$ (note that the latter condition on δ is crucial since we may only use constants from K). We want to ask for the converse, i.e. whether from the equivalence of the amc–structures of level δ, the equivalence of $\mathbf{K}$ and $\mathbf{L}$ over $\mathbf{F}$ may be deduced. However, we do not want to let δ run through all of vK; we will gain a better information if we restrict δ to Δ for some convex subgroup Δ of vK, compensating by a suitable hypothesis on the extensions $\mathbf{L}$ and $\mathbf{F}$ of $\mathbf{K}$.

Given a convex subgroup Δ of vK, there is a unique coarsening v_Δ of the valuation v whose value group is isomorphic to vK/Δ and whose residue field Kv_Δ carries a valuation $\overline{v}_\Delta$ with value group isomorphic to Δ such that v is the composition of v_Δ and $\overline{v}_\Delta$; for details, see e.g. [Z–S].

Consider an arbitrary extension **L** of **K**. There is a unique smallest convex subgroup of vL containing Δ; it is just the convex hull of Δ and will again be denoted by Δ. The coarsening of the valuation v of **L** which corresponds to Δ is a prolongation of v_Δ from K to L and will again be denoted by v_Δ. If the rank of **L** is larger than the rank of **K** then there may exist more than one prolongation, but v_Δ is the finest of them. However, it is uniquely determined in case $L|K$ is algebraic. If (L,v) is henselian, then so is (L,v_Δ); if in addition, $L|K$ is algebraic, then (L,v_Δ) may be viewed as an algebraic extension of some henselization of (K,v_Δ).

An algebraic extension $(k_1,w)|(k,w)$ of henselian fields is called **tame** if k_1 is contained in the **absolute ramification field** of (k,w) which is defined to be the ramification field of the extension $(k^{sep},w)|(k,w)$, where k^{sep} denotes the separable–algebraic closure of k, carrying the unique prolongation of the henselian valuation w of k. It follows that every intermediate extension of a tame extension is again a tame extension: if $(k_1,w)\supset(k_2,w)\supset(k_3,w)\supset(k,w)$ and $(k_1,w)|(k,w)$ is a tame extension, then the same holds for $(k_2,w)|(k_3,w)$. An equivalent characterization of tame extensions is the following: for every finite subextension $(k',w)|(k,w)$ of $(k_1,w)|(k,w)$, the following holds:

(T1) the residue field extension $k'w|kw$ is separable,

(T2) if $p=\mathrm{char}(kw)>0$, then the ramification index $(wk':wk)$ is prime to p,

(T3) the extension is **defectless**, i.e.

$$[k':k]=[k'w:kw]\cdot(wk':wk)\,. \tag{4}$$

Note that in general, the following **fundamental inequality** holds:

$$[k':k]\geq[k'w:kw]\cdot(wk':wk)\,. \tag{5}$$

A henselian field is called a **defectless field** if each of its finite extensions is defectless. An arbitrary valued field is called a **defectless field** if its henselizations are defectless fields. A valued field is called a **tame field** if it is henselian and every algebraic extension is a tame extension. A tame field (k,w) is characterized as being a henselian defectless field with a perfect residue field and a p-divisible value group ($p = \mathrm{char} kw > 0$ or $p = 1$ otherwise). See [K1] for details on tame fields.

A general not necessarily algebraic extension $(k_1, w)|(k, w)$ will be called **pretame** if the following holds:

(P1) the residue field extension $k_1w|kw$ is separable,

(P2) if $p = \mathrm{char}(kw) > 0$, then the order of every torsion element of vk_1/vk is prime to p.

Note that every extension of a tame field is pretame, and that every algebraic pretame extension of a defectless field is tame.

An extension $\mathbf{L}|\mathbf{K}$ of valued fields is called **immediate** if $vL = vK$ and $\overline{L} = \overline{K}$ (if there is no danger of confusion, we will write $\overline{L}, \overline{K}$ instead of Lv, Kv).

To provide the most general setting for our results, let us consider elementary classes $\mathcal{K}$ of valued fields (*always assumed to be nontrivially valued!*) which have the following properties:

(IME) "Immediate extensions are equivalent": if $\mathbf{K}, \mathbf{L}, \mathbf{F} \in \mathcal{K}$ and $\mathbf{L}$ and $\mathbf{F}$ are immediate extensions of $\mathbf{K}$, then $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$,

(RAC) "Relative algebraic closures": if $\mathbf{L} \in \mathcal{K}$, the quotient vL/vK is a torsion group and the extension $\overline{L}|\overline{K}$ is algebraic, then the relative algebraic closure $\mathbf{L}'$ of $\mathbf{K}$ in $\mathbf{L}$ is an element of $\mathcal{K}$, and $\mathbf{L}|\mathbf{L}'$ is immediate.

It is known that the following classes of valued fields have the above properties: tame fields (which includes henselian fields with residue characteristic 0, algebraically closed valued fields, algebraically maximal Kaplansky fields), henselian finitely ramified fields (which includes $\wp$-adically closed fields). All of them are defectless fields. For details, see [K1]. The definitions of algebraically maximal Kaplansky-fields and of finitely ramified fields are given below.

THEOREM 2.1: *Let $\mathcal{K}$ be an elementary class of valued fields which satisfies (IME) and (RAC). Further, let $\mathbf{K}$ be a common valued subfield of the henselian fields $\mathbf{L}$ and $\mathbf{F}$. Suppose that Δ is a convex subgroup of vK such that*

(a) $(L^*, v_\Delta), (F^*, v_\Delta) \in \mathcal{K}$ *for all elementary extensions $\mathbf{L}^*$ and $\mathbf{F}^*$ of $\mathbf{L}$ and F on which v_Δ is nontrivial,*

(b) (K, v_Δ) *is a defectless field,*

(c) (L, v_Δ) *and* (F, v_Δ) *are pretame extensions of* (K, v_Δ).

Then the following statements are equivalent:

(i) $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$,

(ii) $(\mathbf{L}^*)_\Delta \equiv_{K_\Delta} (\mathbf{F}^*)_\Delta$ *for some elementary extensions* $\mathbf{L}^*$, $\mathbf{F}^*$ *of* $\mathbf{L}$ *and* $\mathbf{F}$,

(iii) $\mathbf{L}_\delta \equiv_{\mathbf{K}_\delta} \mathbf{F}_\delta$ *for every* $\delta \in \Delta$.

For $\Delta = \{0\}$ we get the following corollaries:

COROLLARY 2.2: *Let $\mathcal{K}$ be the elementary class of henselian fields of residue characteristic 0. If $\mathbf{L}, \mathbf{F} \in \mathcal{K}$ and $\mathbf{K}$ is a common valued subfield of $\mathbf{L}$ and $\mathbf{F}$ then $\mathbf{L}_0 \equiv_{\mathbf{K}_0} \mathbf{F}_0$ implies $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$. This shows: $\mathcal{K}$ admits quantifier elimination relative to the amc–structures of level 0.*

COROLLARY 2.3: *Assume that $\mathbf{L}, \mathbf{F}$ are tame fields and that $\mathbf{K}$ is a common valued subfield of $\mathbf{L}$ and $\mathbf{F}$. If any henselization of $\mathbf{K}$ is a tame field, then $\mathbf{L}_0 \equiv_{\mathbf{K}_0} \mathbf{F}_0$ implies $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$.*

The latter corollary gives rise to the following question: do there exist functions definable in the theory of tame fields such that in the language of valued fields enriched by these functions, every substructure admits a henselization which is a tame field? If this is true, then the theory of tame fields would admit quantifier elimination relative to amc–structures of level 0 in this enriched language.

The special case considered by Basarab in [B] is the case of valued fields of mixed characteristic, that is, valued fields of characteristic 0 with residue characteristic $p > 0$. Before discussing this case, let us introduce the following notation.

Assume $\mathrm{char} K = 0$ and let p be the **characteristic exponent** of the residue field $\overline{K}$, i.e. $p = \mathrm{char}(\overline{K}) > 0$ or $p = 1$ if $\mathrm{char}(\overline{K}) = 0$. The **canonical decomposition** of the valuation v is defined as follows. Denote by $\Delta_{\mathbf{K}}$ the smallest convex subgroup of vK containing the value vp; note that the value set $\{m \cdot vp \mid m \in \mathbb{N}\}$ is cofinal in $\Delta_{\mathbf{K}}$. We write $\dot{v} := v_{\Delta_{\mathbf{K}}}$; this is called the **coarse valuation** assigned to v. A valued field of characteristic 0 is of mixed characteristic if and only if $\dot{v}$ is coarser than v, i.e. $\Delta_{\mathbf{K}} \neq \{0\}$. Denote by $\dot{\mathbf{K}}$ the valued field $(K, \dot{v})$. The valuation ring $\mathcal{O}_{\dot{\mathbf{K}}}$ of $\dot{\mathbf{K}}$ is characterized as the smallest overring of $\mathcal{O}_{\mathbf{K}}$ in which p becomes a unit, i.e. $\mathcal{O}_{\dot{\mathbf{K}}}$ is the ring of fractions of $\mathcal{O}_{\mathbf{K}}$ with respect to the multiplicatively closed set $\{p^m \mid m \in \mathbb{N}\}$; consequently, the residue field $K\dot{v}$ is of characteristic 0. Note that $\dot{v} = v$ iff $p = 1$, and $\dot{v}$ is trivial iff $\Delta_{\mathbf{K}} = vK$. For $m \in \mathbb{N}$ we write $\mathbf{K}_m$ instead of $\mathbf{K}_{m \cdot vp}$.

Since $\dot{\mathbf{K}}$ has residue characteristic $\mathrm{char} K\dot{v} = 0$, every algebraic extension of a henselization of $\dot{\mathbf{K}}$ is tame. This is immediately seen from the second characterization of tame extensions given above; note that in this case, condition 3) is a consequence of the Lemma of Ostrowski (cf. [RIB], p. 236, Théorème 2). Further, if $\mathbf{L}$ is a henselian field, then $\dot{\mathbf{L}}$ is a henselian field of residue characteristic 0 (possibly trivially valued). Hence, with $\Delta = \Delta_{\mathbf{K}}$ and $\mathcal{K}$ the class of all henselian fields of residue characteristic 0, we obtain from Theorem 2.1 the following corollary:

COROLLARY 2.4: *Let $\mathbf{L}$ and $\mathbf{F}$ be henselian fields of characteristic 0 with residue characteristic p. If $\mathbf{K}$ is a common valued subfield of $\mathbf{L}$ and $\mathbf{F}$, then the condition*

$$\forall m \in \mathbb{N} : \mathbf{L}_m \equiv_{\mathbf{K}_m} \mathbf{F}_m$$

is equivalent to $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$. Hence $\mathcal{K}$ admits quantifier elimination relative to the collection of amc–structures of level $m \cdot vp$, $m \in \mathbb{N}$.

There is a second question that may come to one's mind in view of Corollary 2.3: if it is so that the amc–structures are only powerful enough to deal with tame extensions, then possibly a certain uniqueness property for the "non–tame" algebraic extensions may help us further. This is indeed true, and it is the right moment to look at Kaplansky-fields. Henselian defectless Kaplansky–fields are characterized as tame fields whose residue fields do not admit finite separable extensions of degree divisible by the residue characteristic. The elementary class of henselian defectless Kaplansky–fields has properties (IME) and (RAC). For this and other details on Kaplanky–fields, see [K1]. Note that a Kaplansky–field is henselian and defectless if and only if it is algebraically maximal, i.e. it does not admit finite immediate extensions. The uniqueness property that we will employ in the proof of the next theorem on Kaplansky–fields is the following:

LEMMA 2.5: *If the residue field of the henselian field $\mathbf{K}$ does not admit a finite separable extension of degree divisible by $p = \mathrm{char}\overline{K}$, then the maximal purely wild algebraic extensions of $\mathbf{K}$ are all isomorphic over $\mathbf{K}$.*

An algebraic extension of the henselian field $\mathbf{K}$ is called **purely wild** if it is linearly disjoint from every tame algebraic extension of $\mathbf{K}$. For a proof of this lemma, see [KPR].

THEOREM 2.6: *Let $\mathcal{K}$ be an elementary class of henselian defectless Kaplansky–fields and let $\mathbf{K}$ be a common valued subfield of the henselian fields $\mathbf{L}$ and $\mathbf{F}$. Suppose that Δ is a convex subgroup of vK such that $(L, v_\Delta), (F, v_\Delta) \in \mathcal{K}$. Then again, the statements (i), (ii) and (iii) of Theorem 2.1 are equivalent.*

In particular, henselian defectless Kaplansky–fields admit quantifier elimination relative to the amc–structures of level 0.

Note that this theorem applies even to cases where (K, v_Δ) is not necessarily a defectless field; this is also a consequence of the uniqueness property.

Another question to be asked is whether there are cases in which a reduct of the amc–structures will do the same job. There is a natural reduct of the amc–structures, namely the rings $\mathcal{O}^\delta$.

THEOREM 2.7: *Assume that in addition to the hypothesis of Theorem 2.1, the following condition holds:*

(UNRAM) both extensions $(L, v_\Delta)|(K, v_\Delta)$ and $(F, v_\Delta)|(K, v_\Delta)$ are unramified.

Then the following statements are equivalent:

(i) $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$,

(ii) $\mathcal{O}^\Delta_{\mathbf{L}^*} \equiv_{\mathcal{O}^\Delta_K} \mathcal{O}^\Delta_{\mathbf{F}^*}$ *for some elementary extensions* $\mathbf{L}^*$, $\mathbf{F}^*$ *of* $\mathbf{L}$ *and* $\mathbf{F}$,

(iii) $\mathcal{O}^\delta_{\mathbf{L}} \equiv_{\mathcal{O}^\delta_{\mathbf{K}}} \mathcal{O}^\delta_{\mathbf{F}}$ *for every* $\delta \in \Delta$.

Note that in particular, (UNRAM) holds in the case where $\Delta = vK$.

There is a second natural reduct of the amc–structures, namely the groups G^δ. See Theorem 3.11 and Corollary 3.6 below for the case $\Delta = \{0\}$. Let us discuss here the case where Δ may be a nontrivial convex subgroup of vK; Theorem 1.1 will be deduced from this case. We have to use stronger hypotheses. We will consider elementary classes $\mathcal{K}$ of *nontrivially* valued fields which have the following properties:

(IME$^+$) if $\mathbf{K}, \mathbf{L}, \mathbf{F} \in \mathcal{K}$ and $\mathbf{L}$ and $\mathbf{F}$ are extensions of $\mathbf{K}$, all of them having the same residue field, then $vL \equiv_{vK} vF$ implies $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$, provided that vK is pure in both vL and vF,

(RAC$^+$) if $\mathbf{L} \in \mathcal{K}$ is an extension of $\mathbf{K}$, both having the same residue field, then the relative algebraic closure $\mathbf{L}'$ of $\mathbf{K}$ in $\mathbf{L}$ is an element of $\mathcal{K}$, and vL' is pure in vL.

Again, it is known that the elementary classes of tame fields and of henselian finitely ramified fields have the above properties; see [K1].

THEOREM 2.8: *Let $\mathcal{K}$ be an elementary class of valued fields which satisfies (IME$^+$) and (RAC$^+$). Further, let $\mathbf{K}$ be a common valued subfield of the* *henselian fields* $\mathbf{L}$ *and* $\mathbf{F}$ *with relative algebraic closures* $\mathbf{L}'$ *and* $\mathbf{F}'$ *in* $\mathbf{L}$ *resp.* $\mathbf{F}$. *Suppose that* Δ *is a convex subgroup of* vK *such that*

(a) $(L^*, v_\Delta), (F^*, v_\Delta) \in \mathcal{K}$ *for all elementary extensions* $\mathbf{L}^*$ *and* $\mathbf{F}^*$ *of* $\mathbf{L}$ *and* $\mathbf{F}$ *on which* v_Δ *is nontrivial,*

(b) (L', v_Δ) *and* (F', v_Δ) *are tame extensions of some henselizations of* (K, v_Δ).

Assume further that the following condition holds:

(DENSE) $(Kv_\Delta, \overline{v}_\Delta)$ *is dense in both* $(Lv_\Delta, \overline{v}_\Delta)$ *and* $(Fv_\Delta, \overline{v}_\Delta)$.

Then the following statements are equivalent:

(i) $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$,

(ii) $(G^\Delta_{\mathbf{L}^*}, \mathrm{Pos}_\Delta) \equiv_{(G^\Delta_{\mathbf{K}^*}, \mathrm{Pos}_\Delta)} (G^\Delta_{\mathbf{F}^*}, \mathrm{Pos}_\Delta)$ *and* $L^*v_\Delta = K^*v_\Delta = F^*v_\Delta$ *for some elementary extension* $(\mathbf{L}^*, \mathbf{K}^*, \mathbf{F}^*) = (\mathbf{L}, \mathbf{K}, \mathbf{F})^*$ *of* $(\mathbf{L}, \mathbf{K}, \mathbf{F})$,

(iii) $(G^\delta_{\mathbf{L}}, \mathrm{Pos}_\delta) \equiv_{(G^\delta_{\mathbf{K}}, \mathrm{Pos}_\delta)} (G^\delta_{\mathbf{F}}, \mathrm{Pos}_\delta)$ *for every* $\delta \in \Delta$.

If in addition, the value groups of all valued fields in $\mathcal{K}$ *are divisible, then the following statements are equivalent:*

(I) $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$,

(II) $G^\Delta_{\mathbf{L}^*} \equiv_{G^\Delta_{\mathbf{K}^*}} G^\Delta_{\mathbf{F}^*}$ *and* $L^*v_\Delta = K^*v_\Delta = F^*v_\Delta$ *for some elementary extension* $(\mathbf{L}^*, \mathbf{K}^*, \mathbf{F}^*) = (\mathbf{L}, \mathbf{K}, \mathbf{F})^*$ *of* $(\mathbf{L}, \mathbf{K}, \mathbf{F})$,

(III) $G^\delta_{\mathbf{L}} \equiv_{G^\delta_{\mathbf{K}}} G^\delta_{\mathbf{F}}$ *for every* $\delta \in \Delta$,

(IV) $K \cap L^n = K \cap F^n$ *for every* $n \in \mathbb{N}$.

By "elementary extension $(\mathbf{L}, \mathbf{K}, \mathbf{F})^*$ of $(\mathbf{L}, \mathbf{K}, \mathbf{F})$" we mean an elementary extension of the structure $(\mathbf{L} \supset \mathbf{K} \subset \mathbf{F})$ which may be formalized in a two-sorted language for a pair of valued fields with a predicate for a common subfield. Note that by virtue of condition (DENSE), the equality $L^*v_\Delta = K^*v_\Delta = F^*v_\Delta$ will hold whenever the elementary extension is highly enough saturated.

Instead of requiring that the value groups be divisible it suffices to assume that the value groups of all valued fields in $\mathcal{K}$ are members of one substructure complete elementary class of ordered abelian groups, which is closed under taking pure subgroups (and which may be axiomatized by use of certain constants). Furthermore, in (III) the groups $G^\delta_{\mathbf{L}}$ and $G^\delta_{\mathbf{F}}$ may be replaced by the relative divisible closures of $G^\delta_{\mathbf{K}}$ in $G^\delta_{\mathbf{L}}$ resp. $G^\delta_{\mathbf{F}}$.

A field is called **finitely ramified** if vK has a smallest positive element $v\pi$ and there is a prime p such that vp is a finite multiple of $v\pi$; consequently, $\mathrm{char} K = 0$ and $\mathrm{char} \overline{K} = p$. Then π is called a **prime element of the finitely ramified field K**. For a finitely ramified field $\mathbf{K}$ with prime element π we have $\Delta_{\mathbf{K}} = \mathbb{Z} \cdot v\pi$, and $(Kv_{\Delta_{\mathbf{K}}}, \overline{v}_{\Delta_{\mathbf{K}}})$ is thus a discretely valued field. If $\mathbf{L}$ is finitely ramified and $\mathbf{K}$ is a valued subfield of $\mathbf{L}$, then also $\mathbf{K}$ is finitely ramified. If $\mathbf{K}$ has the same prime element and the same residue field as $\mathbf{L}$, then $(Kv_\Delta, \overline{v}_\Delta)$ is dense in $(Lv_\Delta, \overline{v}_\Delta)$ for $\Delta = \Delta_{\mathbf{K}}$.

COROLLARY 2.9: *Assume that* $\mathbf{L}, \mathbf{F}$ *are henselian finitely ramified fields and that* $\mathbf{K}$ *is a common valued subfield of* $\mathbf{L}$ *and* $\mathbf{F}$*, all of them having the same residue field and the same prime element. Then with* $\Delta = \Delta_{\mathbf{K}}$*, the statements* (i), (ii) *and* (iii) *of Theorem 2.8 are equivalent.*

COROLLARY 2.10: *Assume that in addition to the hypothesis of the preceding corollary,* $v_\Delta L$ *and* $v_\Delta F$ *are divisible (i.e.* vL *and* vF *are* $\mathbb{Z}$*–groups). Then the statements* (I), (II), (III) *and* (IV) *of Theorem 2.8 are equivalent.*

In particular, the theory of $\wp$*–adically closed fields of fixed* p*–rank* d *admits elimination of quantifiers relative to the groups* $G^{m\cdot vp}$, $m \in \mathbb{N}$, *as well as in the language enriched by the power predicates* P_n, $n \in \mathbb{N}$.

In general, it is not possible to restrict the numbers n appearing in condition (IV) and as indices of the power predicates, to a finite set of natural numbers or to a set of natural numbers which are not divisible by a fixed prime. However, there are cases (in particular, of certain "large" Kaplansky–fields), where n has only to range over the powers of one given prime number. These cases will be discussed in the subsequent paper [KK].

3. An embedding lemma for tame algebraic extensions, and generalizations

We will first describe the structure of a finite tame extension $\mathbf{L}|\mathbf{K}$ of henselian fields.

The residue field extension $\overline{L}|\overline{K}$ is finite and separable, hence simple. Let $\overline{c}$ be a generator of it. We choose some monic polynomial $f \in K[X]$ whose reduction modulo v is the irreducible polynomial of $\overline{c}$ over $\overline{K}$. Since the latter is separable, we may use Hensel's Lemma to find a root $c \in L$ of f with residue $\overline{c}$. From general valuation theory it follows that the extension $K(c)|K$ is of the same degree as $\overline{K}(\overline{c})|\overline{K}$ and that $\overline{K(c)} = \overline{K}(\overline{c}) = \overline{L}$. Let us mention that $(K(c), v)$ is the inertia field of our extension $\mathbf{L}|\mathbf{K}$.

Now we have to treat the case of $vL \neq vK$. Let $\alpha \in vL \setminus vK$ and assume that $n \neq 0$ is the minimal natural number such that $n\alpha \in vK$. If $a \in L$ with

$va = \alpha$, then $va^n \in vK$ and thus there is some $b \in K$ with $v(ba^n) = 0$. Then the v–residue $\overline{ba^n} \in Lv = K(c)v$ is not zero, hence there is some $h \in K[X]$ with $\overline{a^{-n}b^{-1}h(c)} = 1$. By the minimality of n and condition (T2) for tame extensions, n is prime to p if $\mathrm{char}(Kv) = p > 0$. Hence in the henselian field $\mathbf{L}$, we may use Hensel's Lemma to deduce the existence of some element $a_0 \in L$ which satisfies $a_0^n = a^{-n}b^{-1}h(c)$; putting $d = aa_0 \in L$ we get $bd^n = h(c)$. Note that we may choose h with v–integral coefficients since it only has to satisfy $\overline{h}(\overline{c}) = \overline{a^n b} \in \overline{K}(\overline{c})$ where $\overline{h}$ denotes the reduction of h modulo v.

Since $L|K$ is a finite extension, the group vL/vK is a finite torsion group, say

$$vL/vK = \mathbb{Z} \cdot (\alpha_1 + vK) \times \cdots \times \mathbb{Z} \cdot (\alpha_r + vK) \tag{6}$$

where every α_i has finite order, say n_i. Using the above procedure, for $1 \leq i \leq r$ we choose elements

- $d_i \in L$ with $b_i d_i^{n_i} = h_i(c)$, where
- $b_i \in K$ with $v(b_i) = -n_i\alpha_i$
- $h_i \in \mathcal{O}_{\mathbf{K}}[X]$ with $vh_i(c) = 0$.

Then $vL = vK(c, d_1, \ldots, d_r)$, and since $K(c) \subset K(c, d_1, \ldots, d_r) \subset L$ and $K(c)v = Lv$, we also have $Lv = K(c, d_1, \ldots, d_r)v$. From condition (T3) on tame extensions it follows that $L = K(c, d_1, \ldots, d_r)$. On the other hand,

$$\begin{aligned}[L : K(c)] &\geq (vL : vK(c)) = (vL : vK) = n_1 \cdots n_r \\ &\geq [K(c, d_1, \ldots, d_r) : K(c)] = [L : K(c)]\,,\end{aligned}$$

whence $[L : K(c)] = n_1 \cdots n_r$ which shows that the extensions

$$K(c, d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_r)|K(c) \quad \text{and} \quad K(c, d_i)|K(c)$$

are linearly disjoint for every i, $1 \leq i \leq r$. Hence, if $\mathbf{K} \subset \mathbf{F}$ and $z, t_1, \ldots, t_r \in F$ such that $f(z) = 0$ and $b_i t_i^{n_i} = h_i(z)$, then

$$(c, d_1, \ldots, d_r) \mapsto (z, t_1, \ldots, t_r)$$

induces an embedding of L into F over K. Since $\mathbf{K}$ is henselian, this is valuation preserving, i.e. an embedding of $\mathbf{L}$ into $\mathbf{F}$ over $\mathbf{K}$.

Let us note the following special cases:

- if $\overline{L} = \overline{K}$, then we may take $\overline{c}$, c and all $h_i(c)$ to be equal to 1,

– if $\mathbf{L}|\mathbf{K}$ is unramified, then we may set $r = 0$.

Using the "normal form" for finite tame extensions that we have now introduced, we will prove the main embedding lemma for tame algebraic extensions:

LEMMA 3.1: *Let* $\mathbf{K}$ *be an arbitrary valued field,* $\mathbf{L}$ *a tame algebraic extension of some henselization of* $\mathbf{K}$ *and* $\mathbf{F}$ *an arbitrary henselian extension of* $\mathbf{K}$*. If* $\mathbf{L}$ *is embeddable into* $\mathbf{F}$ *over* $\mathbf{K}$*, then* $\mathbf{L}_0$ *is embeddable into* $\mathbf{F}_0$ *over* $\mathbf{K}_0$*. Conversely, every embedding* τ *of* $\mathbf{L}_0$ *into* $\mathbf{F}_0$ *over* $\mathbf{K}_0$ *may be pulled back to an embedding of* $\mathbf{L}$ *into* $\mathbf{F}$ *over* $\mathbf{K}$ *which induces* τ*.*

If in addition, $\mathbf{L}|\mathbf{K}$ *is unramified, then the same works for every embedding of* $\overline{L}$ *into* $\overline{F}$ *over* $\overline{K}$*. If on the other hand* $\overline{L} = \overline{K}$*, then the same works for every embedding of* $G_{\mathbf{L}}$ *into* $G_{\mathbf{F}}$ *over* $G_{\mathbf{K}}$*.*

Proof: The proof of the first statement is straightforward and thus left to the reader. Let now be given an embedding τ of $\mathbf{L}_0$ into $\mathbf{F}_0$ over $\mathbf{K}_0$.

Since both $\mathbf{L}$ and $\mathbf{F}$ are assumed to be henselian, they both contain henselizations of $\mathbf{K}$. By the uniqueness property of henselizations, these are isomorphic over $\mathbf{K}$ and we may identify them. This henselization has the same amc–structure of level 0 as $\mathbf{K}$: for every a in a henselization of $\mathbf{K}$ there is some $a' \in K$ such that $v(a - a') > va$, so a and a' have the same images under π_0 and π_0^*. Hence it suffices to prove our lemma under the additional hypothesis that $\mathbf{K}$ be henselian.

The following compactness argument for algebraic extensions is well known: *Since* $L|K$ *is algebraic, we have* $L = \bigcup_{i\in I} K_i$ *where* $K_i|K$ *runs through all finite subextensions of* $L|K$*. If every* K_i *admits an embedding* ι_i *into* F *over* K*, then there is an embedding* ι *of* L *into* F *over* K *and a subset* $J \subset I$ *with* $L = \bigcup_{j\in J} K_j$ *and* $\forall j \in J$: $\iota|_{K_j} = \iota_j$*. In particular, if all* ι_i *are valuation preserving, then so is* ι*, and if all are pullbacks of the respective restrictions of* τ*, then* ι *is a pullback of* τ*.*

So we have to prove our lemma only in the case of $\mathbf{L}|\mathbf{K}$ a finite extension of henselian fields. Let $\mathbf{L}|\mathbf{K}$ be given as described above. By the remark preceding our lemma, it suffices to find an image in $\mathbf{F}$ for the tuple $(c, d_1, \ldots, d_r)$ in order to obtain an embedding of $\mathbf{L}$ into $\mathbf{F}$ over $\mathbf{K}$. This tuple satisfies

$$\overline{f}(\overline{c}) = 0 \quad\text{and}\quad \bigwedge_{1\le i\le r} \Theta_0(\overline{h_i}(\overline{c})\,,\, \tilde{b}_i\tilde{d}_i^{\,n_i}) \wedge \overline{h_i}(\overline{c}) \neq 0$$

where $\tilde{b}_i = \pi_0^* b_i$ and $\tilde{d}_i = \pi_0^* d_i$. Now τ sends $\overline{c}$ to some element $x \in \overline{F}$ and every

$\tilde{d}_i$ to some $y_i \in G_{\mathbf{F}}$ which satisfy

$$\overline{f}(x) = 0 \quad \text{and} \quad \bigwedge_{1 \le i \le r} \Theta_0(\overline{h_i}(x)\,,\, \tilde{b}_i y_i^{n_i}) \wedge \overline{h_i}(x) \neq 0$$

The polynomial $\overline{f}$ being irreducible and separable over $\overline{K}$, the zero x is simple and thus gives rise to a zero $z \in F$ of f with residue x by virtue of Hensel's Lemma.

Now let $i \in \{1, \ldots, r\}$. We choose $\eta_i \in F$ such that $\pi_0^* \eta_i = y_i$. Since $\overline{h_i}(x) \neq 0$, the relation $\Theta_0(\overline{h_i}(x), \tilde{b}_i y_i^{n_i})$ is equivalent to $\vartheta_0 \overline{h_i}(x) = \tilde{b}_i y_i^{n_i}$ which in turn gives

$$\pi_0^* h_i(z) = \vartheta_0 \pi_0 h_i(z) = \vartheta_0 \overline{h_i}(x) = \tilde{b}_i y_i^{n_i} = \pi_0^* b_i \eta_i^{n_i} \;,$$

that is, $h_i(z) b_i^{-1} \eta_i^{-n_i} \equiv 1 \bmod \mathcal{M}_{\mathbf{F}}$. So the polynomial

$$X^{n_i} - h_i(z) b_i^{-1} \eta_i^{-n_i} \in \mathcal{O}_F[X] \tag{7}$$

reduces modulo v to the polynomial $X^{n_i} - 1$ which admits 1 as a simple root since n_i is not divisible by the characteristic of $\overline{K}$. By virtue of Hensel's Lemma, the polynomial (7) admits a root η_i' in the henselian field $\mathbf{F}$. Putting $t_i := \eta_i' \eta_i \in F$, we obtain $b_i t_i^{n_i} = h_i(z)$. Consequently, the assignment $(c, d_1, \ldots, d_r) \mapsto (z, t_1, \ldots, t_r)$ induces an embedding of $\mathbf{L}$ into $\mathbf{F}$ over $\mathbf{K}$.

We still have to show that it is a pullback of τ. But this will follow if we are able to show that the assignment $(\overline{c}, \tilde{d}_1, \ldots, \tilde{d}_r) \mapsto (x, y_1, \ldots, y_r)$ determines the embedding of $\mathbf{L}_0$ into $\mathbf{F}_0$ over $\mathbf{K}_0$ uniquely. Since $\overline{c}$ generates $\overline{L}$ over $\overline{K}$, it just remains to show that the elements $\tilde{d}_1, \ldots, \tilde{d}_r$ generate $G_{\mathbf{L}}$ over the group compositum $G_{\mathbf{K}}.\vartheta_0 \overline{L}$. Given an element $a \in L$, our choice of the d_i implies that there exist integers $m_1, \ldots, m_r$, an element $d' \in K$ and an element $g(c) \in \mathcal{O}_K[c]$ of value 0 such that the value of $a^{-1} d_1^{m_1} \cdots d_r^{m_r} d' g(c)$ is 0 and its residue is 1. Hence

$$\pi_0^* a = \tilde{d}_1^{m_1} \cdot \ldots \cdot \tilde{d}_r^{m_r} \cdot \pi_0^* d' \cdot \vartheta_0 \overline{g}(\overline{c})$$

with $\pi_0^* d' \in G_{\mathbf{K}}$ and $\vartheta_0 \overline{g}(\overline{c}) \in \vartheta_0 \overline{L}$. This concludes our proof. (The special cases mentioned in the lemma are shown by straightforward modifications of this proof.) ∎

From this proof, we may extract one more interesting case, namely the case where the relation Θ_0 may be omitted. We see from the proof that it is indeed superfluous if all $h_i(c)$ can be chosen to be an element of K. But this means that in $\mathbf{L}$ there exists a subfield $\mathbf{C}$ which has the same value group as $\mathbf{L}$, the same residue field as $\mathbf{K}$ and is a field complement of the inertia field $\mathbf{L}^i$ of $\mathbf{L}|\mathbf{K}$ over the henselization $\mathbf{K}^h$ of $\mathbf{K}$ in $\mathbf{L}$. This means, $\mathbf{C}$ is linearly disjoint from $\mathbf{L}^i$ over $\mathbf{K}^h$ and the compositum $\mathbf{L}^i.\mathbf{C}$ equals $\mathbf{L}$. Conversely, one can show that every field complement $\mathbf{C}$ of the inertia field $\mathbf{L}^i$ in $\mathbf{L}$ over $\mathbf{K}^h$ has the property $vC = vL$ and $\overline{C} = \overline{K}$. Since $\mathbf{L}|\mathbf{K}^h$ is supposed to be a tame algebraic extension, the same is true for the subextension $\mathbf{C}|\mathbf{K}^h$.

LEMMA 3.2: *Let $\mathbf{K}$ be an arbitrary valued field, $\mathbf{C}$ a tame algebraic extension of some henselization $\mathbf{K}^h$ of $\mathbf{K}$ such that $\overline{C} = \overline{K}$. Then $\mathbf{C}$ is generated over $\mathbf{K}^h$ by its subset*

$$R = \bigcup_{n\in\mathbb{N}} \{x \in C \mid x^n \in K\} .$$

of radicals over K, i.e. $\mathbf{C} = (K^h(R), v)$ (which is equal to the henselization of $(K(R), v)$ inside of $\mathbf{C}$).

Proof: Let $\alpha \in vC \setminus vK$ and $n \in \mathbb{N}\setminus\{0\}$ minimal with $n\alpha \in vK$. Choose $a \in C$ with $va = \alpha$ and $c \in K$ such that $v(a^n - c) > 0$ (which is possible since $va^n \in vK$ and $\overline{C} = \overline{K}$). By our hypothesis it follows that $\mathbf{C}|\mathbf{K}^h$ is a tame extension, so n (being minimal with $n\alpha \in vK$) is not divisible by the residue characteristic. Hence, by virtue of Hensel's Lemma there exists an element $a_0 \in C$ of value 0 such that $a_0^n = a^{-n}c$. Replacing a by aa_0, we obtain an element $a \in C$ of value α which satisfies $a^n \in K$.

Let R be the collection of all radicals a obtained in this way for all $\alpha \in vC \setminus vK$. Then $(K^h(R), v)$ has the same value group as $\mathbf{C}$. Since $\overline{C} = \overline{K}$, it also has the same residue field as $\mathbf{C}$. As a part of a tame extension, $\mathbf{C}|(K^h(R), v)$ is tame. Since it is immediate, it must be trivial: $C = K^h(R)$. ∎

We may now deduce the following lemma from the proof of Lemma 3.1:

LEMMA 3.3: *Let the hypothesis be as in Lemma 3.1 and assume in addition that there exists a field complement $\mathbf{C}$ of the inertia field $\mathbf{L}^i$ in $\mathbf{L}$ over $\mathbf{K}^h$.*

(a) *For all embeddings ρ of $\overline{L}$ into $\overline{F}$ over $\overline{K}$ and σ of $G_{\mathbf{L}}$ into $G_{\mathbf{F}}$ over $G_{\mathbf{K}}$, there exists an embedding of $\mathbf{L}$ into $\mathbf{F}$ over $\mathbf{K}$ which induces ρ and σ.*
(b) *If $\forall n \in \mathbb{N} : K \cap L^n \subset K \cap F^n$, then for every embedding ρ of $\overline{L}$ into $\overline{F}$ over $\overline{K}$ there exists an embedding of $\mathbf{L}$ into $\mathbf{F}$ over $\mathbf{K}$ which induces ρ.*

Note: if also $\mathbf{F}$ is a tame algebraic extension of some henselization of $\mathbf{K}$ which admits a field complement of its inertia field, then the embedding σ in (a) may be replaced by an embedding of vL into vF over vK. The proof is left to the reader.

Two algebraic extensions of $\mathbf{K}$ are isomorphic over $\mathbf{K}$ if they can be embedded into each other over $\mathbf{K}$. Hence we get the following theorem as an immediate corollary to Lemma 3.1 and Lemma 3.3. It may be seen as a classification of tame algebraic extensions relative to their amc–structures of level 0.

THEOREM 3.4: *Let $\mathbf{K}$ be an arbitrary valued field and $\mathbf{L}$, $\mathbf{F}$ tame algebraic extensions of some henselizations of $\mathbf{K}$. Then $\mathbf{L}$ and $\mathbf{F}$ are isomorphic over $\mathbf{K}$ if and only if their amc–structures of level 0 are isomorphic over $\mathbf{K}_0$. Under the additional hypothesis of Lemma 3.3, the isomorphism will follow already from*

(1) *an isomorphism of the amc–structures without the Θ_0–relation, or*

(2) *an isomorphism $\overline{L} \cong \overline{F}$ over $\overline{K}$ together with the condition* $\forall n \in \mathbb{N} : K \cap L^n = K \cap F^n$.

Moreover, if $\mathbf{L}|\mathbf{K}$ and $\mathbf{F}|\mathbf{K}$ are unramified, then the isomorphism follows already if $\overline{L}$ and $\overline{F}$ are isomorphic over $\overline{K}$. If on the other hand, $\mathbf{L}$, $\mathbf{F}$ and $\mathbf{K}$ have the same residue field, then the isomorphism follows already if $G_{\mathbf{L}}$ and $G_{\mathbf{F}}$ are isomorphic over $G_{\mathbf{K}}$, or if $\forall n \in \mathbb{N} : K \cap L^n = K \cap F^n$.

In all preceding conditions, the isomorphism $G_{\mathbf{L}} \cong G_{\mathbf{F}}$ over $G_{\mathbf{K}}$ may be replaced by an isomorphism $vL \cong vF$ over vK, if the hypothesis of Lemma 3.3 applies to both $\mathbf{L}$ and $\mathbf{F}$.

Similarly, we may use Lemma 3.3 to prove

COROLLARY 3.5: *Let $\mathcal{K}$ be an elementary class of valued fields which satisfies (IME$^+$) and (RAC$^+$). Let $\mathbf{L}, \mathbf{F} \in \mathcal{K}$ and assume that $\mathbf{K}$ is a common subfield of $\mathbf{L}$ and $\mathbf{F}$ such that the relative algebraic closures of $\mathbf{K}$ in $\mathbf{L}$ and $\mathbf{F}$ are tame extensions of some henselizations of $\mathbf{K}$. Assume further that $\mathbf{L}$, $\mathbf{K}$ and $\mathbf{F}$ have the same residue fields. Then the following statements are equivalent:*

(i) $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$,

(ii) $(G_{\mathbf{L}}, \mathrm{Pos}) \equiv_{(G_{\mathbf{K}}, \mathrm{Pos})} (G_{\mathbf{F}}, \mathrm{Pos})$,

(iii) $G_{\mathbf{L}} \equiv_{G_{\mathbf{K}}} G_{\mathbf{F}}$ *and* $vL \equiv_{vK} vF$,

(iv) $\forall n \in \mathbb{N}$: $K \cap L^n = K \cap F^n$ *and* $vL \equiv_{vK} vF$.

Proof: The proof that (i) implies (ii), (iii) and (iv) is straightforward and thus left to the reader. By (RAC$^+$), the relative algebraic closures $\mathbf{L}'$ and $\mathbf{F}'$ of $\mathbf{K}$ in $\mathbf{L}$ resp. $\mathbf{F}$ are members of $\mathcal{K}$, and vL' is pure in vL as well as vF' is pure in vF.

By virtue of the preceding theorem or Lemma 3.3, $\forall n \in \mathbb{N} : K \cap L'^n = K \cap F'^n$ would imply $\mathbf{L}' \cong \mathbf{F}'$ over $\mathbf{K}$. But $K \cap L'^n = K \cap F'^n$ follows from $K \cap L^n = K \cap F^n$ since $K \cap L'^n = K \cap L^n$ and $K \cap F'^n = K \cap F^n$.

$G_{\mathbf{L}} \equiv_{G_{\mathbf{K}}} G_{\mathbf{F}}$ implies that the relative divisible closures $G'_{\mathbf{L}}$ and $G'_{\mathbf{F}}$ of $G_{\mathbf{K}}$ in $G_{\mathbf{L}}$ resp. $G_{\mathbf{F}}$ are isomorphic over $G_{\mathbf{K}}$. Since $\mathbf{L}'|\mathbf{K}$ and $\mathbf{F}'|\mathbf{K}$ are algebraic, vL'/vK and vF'/vK are torsion groups. In view of $\overline{L'} = \overline{K} = \overline{F'}$ and (2), this shows that $G_{\mathbf{L}'}/G_{\mathbf{K}}$ and $G_{\mathbf{F}'}/G_{\mathbf{K}}$ are torsion groups. Hence, $G_{\mathbf{L}'}, G_{\mathbf{F}'}$ are contained in $G'_{\mathbf{L}}$ resp. $G'_{\mathbf{F}}$. The isomorphism of the latter groups induces embeddings of $G_{\mathbf{L}'}$ into $G_{\mathbf{F}}$ and of $G_{\mathbf{F}'}$ into $G_{\mathbf{L}}$, both over $G_{\mathbf{K}}$. By virtue of Lemma 3.3, this again implies $\mathbf{L}' \cong \mathbf{F}'$ over $\mathbf{K}$.

We have shown that each of the statements (ii), (iii) and (iv) implies $\mathbf{L}' \cong \mathbf{F}'$ over $\mathbf{K}$. We may thus identify $\mathbf{L}'$ and $\mathbf{F}'$. Since $\mathbf{L}'|\mathbf{K}$ is algebraic, vL'/vK is a torsion group. Hence $vL \equiv_{vK} vF$ implies $vL \equiv_{vL'} vF$. From our hypothesis that $\mathcal{K}$ has the property (IME$^+$) it now follows that $\mathbf{L} \equiv_{\mathbf{L}'} \mathbf{F}$ and hence also $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$.

It remains to show that $(G_{\mathbf{L}}, \mathrm{Pos}) \equiv_{(G_{\mathbf{K}}, \mathrm{Pos})} (G_{\mathbf{F}}, \mathrm{Pos})$ implies $vL \equiv_{vK} vF$. But (3) shows that every formula in vK may be encoded by a formula in $(G_{\mathbf{K}}, \mathrm{Pos})$. This concludes our proof. ∎

An application of this corollary to tame fields reads as follows. Note that the elementary class of divisible ordered abelian groups is substructure complete (i.e., admits quantifier elimination); this was proved by Robinson and Zakon [R–Z].

COROLLARY 3.6: *Assume that* $\mathbf{L}, \mathbf{F}$ *are tame fields and that* $\mathbf{K}$ *is a common valued subfield of* $\mathbf{L}$ *and* $\mathbf{F}$*, all of them having the same residue field. If any henselization of* $\mathbf{K}$ *is a tame field, then* $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$ *is equivalent to the condition* $(G_{\mathbf{L}}, \mathrm{Pos}) \equiv_{(G_{\mathbf{K}}, \mathrm{Pos})} (G_{\mathbf{F}}, \mathrm{Pos})$ *(and to* $\forall n \in \mathbb{N}$: $K \cap L^n = K \cap F^n$ *in case* vL *and* vF *are divisible).*

For the case of $\overline{L}, \overline{K}, \overline{F}$ not all being equal, we need a generalization of Lemma 3.1. Suppose that $\mathbf{K}$ is a defectless field, $\mathbf{F}$ a henselian extension field of $\mathbf{K}$ and $\mathbf{L}|\mathbf{K}$ a pretame extension which admits a valuation transcendence basis $\mathcal{T}$. That is, $\mathcal{T}$ is a transcendence basis of $L|K$ of the form

(8) $\begin{cases} \mathcal{T} = \{x_i, y_j \mid i \in I,\ j \in J\} \text{ such that:} \\ \text{(a) the values } vx_i,\ i \in I \text{ form a maximal system of values} \\ \text{in } vL \text{ which are rationally independent over } vK, \text{ and} \\ \text{(b) the residues } \overline{y_j},\ j \in J \text{ form a transcendence basis of } \overline{L}|\overline{K}. \end{cases}$

Suppose that τ is an embedding of $\mathbf{L}_0$ into $\mathbf{F}_0$ over $\mathbf{K}_0$. Then $\overline{L}$ is embeddable into $\overline{F}$ over $\overline{K}$. But also vL is embeddable into vF over vK since $vL \cong G_{\mathbf{L}}/\vartheta_0\overline{L}^{\times}$ and the order relation on vK is induced by Pos. Denote these embeddings by σ and ρ and choose a set $\mathcal{T}' = \{x_i', y_j' \mid i \in I, j \in J\} \subset F$ such that $vx_i' = \rho v x_i$ for all $i \in I$, and $\overline{y_j'} = \sigma\overline{y_j}$ for all $j \in J$. Then $\mathcal{T}'$ is a valuation transcendence basis of the subextension $(K(\mathcal{T}), v)|\mathbf{K}$ of $\mathbf{F}|\mathbf{K}$, and the assignment $x_i \mapsto x_i'$, $i \in I$, $y_j \mapsto y_j'$, $j \in J$, induces a valuation preserving isomorphism from $(K(\mathcal{T}), v)$ onto $(K(\mathcal{T}'), v)$ over $\mathbf{K}$ (more precisely spoken, this isomorphism induces the embedding ρ on the value groups and the embedding σ on the residue fields – cf. [BOU], chapter VI, §10.3, Theorem 1). This isomorphism is even a pullback of the isomorphism of the respective amc–structures of level 0 which is a restriction of τ. Similarly as in the proof of Lemma 3.1, this is shown by proving that the residues $\overline{y}_j$, $j \in J$, generate $\overline{K(\mathcal{T})}$ over $\overline{K}$ and that the elements $\pi_0^* x_i$, $i \in I$, generate $G_{(K(\mathcal{T}),v)}$ over the compositum $G_{\mathbf{K}} \,.\, \vartheta_0\overline{K(\mathcal{T})}^{\times}$.

Hence we may identify $K(\mathcal{T})$ and $K(\mathcal{T}')$ as a common valued subfield of $\mathbf{L}$ and $\mathbf{F}$. We may now apply Lemma 3.1 to get:

LEMMA 3.7: *Let $\mathbf{K}$ be a common subfield of the henselian fields $\mathbf{L}$ and $\mathbf{F}$. Assume that $\mathbf{L}$ admits a valuation transcendence basis $\mathcal{T}$ such that $\mathbf{L}$ itself is a tame extension of some henselization $(K(\mathcal{T}), v)^h$. Then for every embedding τ of $\mathbf{L}_0$ into $\mathbf{F}_0$ over $\mathbf{K}_0$ there is an embedding of $\mathbf{L}$ into $\mathbf{F}$ over $\mathbf{K}$ which induces τ.*

The special cases mentioned in Lemma 3.1 go through as follows. If $\mathbf{L}|\mathbf{K}$ is unramified then again, an embedding of $\overline{L}$ into $\overline{F}$ over $\overline{K}$ will suffice. On the other hand, if $\overline{L} = \overline{K}$ then a simple embedding of $G_{\mathbf{L}}$ into $G_{\mathbf{F}}$ over $G_{\mathbf{K}}$ may not suffice. We have seen above that in the ramified case, an embedding ρ of vL into vF over vK is needed. But in view of (3), this will be induced by an embedding of $(G_{\mathbf{L}}, \mathrm{Pos})$ into $(G_{\mathbf{F}}, \mathrm{Pos})$ over $(G_{\mathbf{K}}, \mathrm{Pos})$; hence in the case $\overline{L} = \overline{K}$, such an embedding will suffice. For the following considerations, let us keep these special cases in mind.

In the sequel, let us assume in addition that $\mathbf{K}$ is a defectless field and that $\mathbf{L}$ is the henselization of a finitely generated extension $\mathbf{K}'$ of $\mathbf{K}$. Then $\mathbf{K}'$ is a finite extension of $K(\mathcal{T})$ for every transcendence basis $\mathcal{T}$ of $\mathbf{K}'|\mathbf{K}$. From general valuation theory it follows that also $vL = vK'$ is finitely generated over vK and that $\overline{L} = \overline{K'}$ is finitely generated over $\overline{K}$. Since $\overline{L}|\overline{K}$ is separable by assumption, we may choose the elements y_j such that their residues $\overline{y}_j$ form a separating transcendence basis of $\overline{L}|\overline{K}$. Similarly, since the order of every torsion element of vL/vK is prime to $p = \mathrm{char}\,\overline{K}$, we may choose the elements x_i such that p does not divide $(vL : vK(\mathcal{T}))$. Since $\mathbf{K}$ is assumed to be a defectless field, the same is true for $(K(\mathcal{T}), v)$ by virtue of [K1], Theorem 3.1. This shows that for $(K(\mathcal{T}), v)^h$ the henselization inside of $\mathbf{L}$, the finite extension $\mathbf{L}|(K(\mathcal{T}), v)^h$ is tame.

Let $\mathbf{L}|\mathbf{K}$ be any pretame extension which admits a valuation transcendence basis $\mathcal{T}$. Then every finitely generated subextension is contained in a henselization of a finitely generated pretame subextension which admits a finite subset $\mathcal{T}_0 \subset \mathcal{T}$ as its valuation transcendence basis. This in turn is a tame algebraic extension of some henselization $(K(\mathcal{T}_0), v)^h$, as we have just shown. Let τ be an embedding of $\mathbf{L}_0$ into $\mathbf{F}_0$ over $\mathbf{K}_0$, and assume in addition that $\mathbf{F}$ is $|L|^+$–saturated. Lemma 3.7 then yields that every finitely generated subextension of $\mathbf{L}|\mathbf{K}$ may be embedded over $\mathbf{K}$ into $\mathbf{F}$. So by general model theory (cf. [P], Korollar 2.19), there is an embedding of $\mathbf{L}$ into $\mathbf{F}$ over $\mathbf{K}$. One may even assume that this embedding induces τ; this is shown as in the proof of Lemma 8.2 of [K1] by introducing additional predicates which in some sense represent the embedding τ.

Now we have to deal with the case where $\mathbf{L}|\mathbf{K}$ does not admit a valuation transcendence basis. Nevertheless, there are subfields which are maximal in having a valuation transcendence basis. We just have to take $\mathcal{T}$ as in (8) and form the subfield $\mathbf{L}'' := (K(\mathcal{T}), v)$ of $\mathbf{L}$. We observe that by definition of $\mathcal{T}$, the quotient vL/vL'' is a torsion group and $\overline{L}|\overline{L''}$ is algebraic. Consequently, if $\mathbf{L}$ is a member of some elementary class $\mathcal{K}$ which has the (RAC) property, then we may take $\mathbf{L}'$ to be the relative algebraic closure of $\mathbf{L}''$ in $\mathbf{L}$ to obtain an intermediate field $\mathbf{L}' \in \mathcal{K}$ which admits a valuation transcendence basis and satisfies that $\mathbf{L}|\mathbf{L}'$ is an immediate extension. We have thereby proved the following lemma:

LEMMA 3.8: *Let $\mathbf{K}$ be a common defectless subfield of the henselian fields $\mathbf{L}$ and $\mathbf{F}$ such that $\mathbf{L}|\mathbf{K}$ is a pretame extension. Assume that $\mathbf{L}$ is a member of an elementary class $\mathcal{K}$ of valued fields which has the (RAC) property. Then there exists a subfield $\mathbf{L}' \in \mathcal{K}$ of $\mathbf{L}$ which admits a valuation transcendence basis and such that $\mathbf{L}|\mathbf{L}'$ is immediate. Moreover, if $\mathbf{F}$ is an $|L|^+$–saturated extension of $\mathbf{K}$, then for every embedding τ of $\mathbf{L}_0$ into $\mathbf{F}_0$ over $\mathbf{K}_0$ there is an embedding of $\mathbf{L}$ into $\mathbf{F}$ over $\mathbf{K}$ which induces τ.*

We have remarked in the introduction that every formula φ in the language of amc–structures of level δ can be encoded by a formula φ_δ in the language of valued fields augmented by a constant symbol for an arbitrary element in K of value δ. Thus, by general model theory one may derive the following

LEMMA 3.9: *If $\mathbf{K}^*$ is an elementary extension of $\mathbf{K}$ and a special model of cardinality $\kappa > \operatorname{card}\mathbf{K}$, then $(\mathbf{K}^*)_\delta$ is an elementary extension of $\mathbf{K}_\delta$ which is also a special model of cardinality κ.*

By virtue of this lemma and a back and forth argument (similar to the proof of the uniqueness of special models, cf. [C–K], Thm. 5.1.17), Lemma 3.8 leads to

LEMMA 3.10: *Let $\mathcal{K}$ be an elementary class of henselian fields which has the (RAC) property, and let $\mathbf{K}$ be a common defectless subfield of $\mathbf{L},\mathbf{F} \in \mathcal{K}$ such that $\mathbf{L}|\mathbf{K}$ and $\mathbf{F}|\mathbf{K}$ are pretame extensions. If $\mathbf{L}_0 \equiv_{\mathbf{K}_0} \mathbf{F}_0$, then there exist elementary extensions $\mathbf{L}^*$ and $\mathbf{F}^*$ of $\mathbf{L}$ and $\mathbf{F}$ which contain relatively algebraically closed subfields $\mathbf{L}'$ resp. $\mathbf{F}'$ such that*

(a) $\mathbf{L}',\mathbf{F}' \in \mathcal{K}$,

(b) *$\mathbf{L}'$ and $\mathbf{F}'$ are isomorphic over $\mathbf{K}$,*

(c) *$\mathbf{L}^*|\mathbf{L}'$ and $\mathbf{F}^*|\mathbf{F}'$ are immediate extensions,*

(d) $(\mathbf{L}^*)_0 \cong_{(\mathbf{K}^*)_0} (\mathbf{F}^*)_0$.

Again, the special cases go through. If $\mathbf{L}|\mathbf{K}$ and $\mathbf{F}|\mathbf{K}$ are unramified, then $\mathbf{L}_0 \equiv_{\mathbf{K}_0} \mathbf{F}_0$ may be replaced by $\overline{L} \equiv_{\overline{K}} \overline{F}$. If $\mathbf{L}$, $\mathbf{K}$ and $\mathbf{F}$ have the same residue fields, then $\mathbf{L}_0 \equiv_{\mathbf{K}_0} \mathbf{F}_0$ may be replaced by $(G_{\mathbf{L}},\mathrm{Pos}) \equiv_{(G_{\mathbf{K}},\mathrm{Pos})} (G_{\mathbf{F}},\mathrm{Pos})$.

From what we have shown above, we obtain the following special case of Theorem 2.1:

THEOREM 3.11: *Let $\mathcal{K}$ be an elementary class of henselian fields which has the properties (IME) and (RAC). Further, let $\mathbf{K}$ be a common defectless subfield of $\mathbf{L},\mathbf{F} \in \mathcal{K}$ such that $\mathbf{L}|\mathbf{K}$ and $\mathbf{F}|\mathbf{K}$ are pretame extensions. Then $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$ is equivalent to $\mathbf{L}_0 \equiv_{\mathbf{K}_0} \mathbf{F}_0$.*

If $\mathbf{L}|\mathbf{K}$ and $\mathbf{F}|\mathbf{K}$ are unramified, then $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$ is equivalent to $\overline{L} \equiv_{\overline{K}} \overline{F}$. If on the other hand, $\mathbf{L}$, $\mathbf{F}$ and $\mathbf{K}$ have the same residue field, then $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$ is equivalent to $(G_{\mathbf{L}},\mathrm{Pos}) \equiv_{(G_{\mathbf{K}},\mathrm{Pos})} (G_{\mathbf{F}},\mathrm{Pos})$.

In the sequel, we will adapt these results for the case of Kaplansky–fields. To do this, we need the following lemma.

LEMMA 3.12: *Let $\mathbf{L}$ and $\mathbf{F}$ be two henselian defectless Kaplansky–fields and $\mathbf{K}$ a common henselian subfield of them. Assume that both vL/vK and vF/vK are* *torsion groups and both* $\overline{L}|\overline{K}$ *and* $\overline{F}|\overline{K}$ *are algebraic extensions. If* $\mathbf{K}$ *does not admit any nontrivial tame algebraic extension inside of* $\mathbf{L}$ *or* $\mathbf{F}$, *then the relative algebraic closures of* $\mathbf{K}$ *in* $\mathbf{L}$ *and* $\mathbf{F}$ *are isomorphic over* $\mathbf{K}$.

Proof: From the hypothesis and the fact that henselian defectless Kaplansky–fields have the (RAC)–property, we deduce that the relative algebraic closures of $\mathbf{K}$ in $\mathbf{L}$ and $\mathbf{F}$ are both henselian defectless Kaplansky–fields, i.e. tame fields whose residue fields do not admit a finite separable extension of degree divisible by $p = \mathrm{char}\overline{K}$. Hence they are both maximal purely wild algebraic extensions of $\mathbf{K}$. Moreover, the residue field of $\mathbf{K}$ does not admit a finite separable extension of degree divisible by p (since otherwise, this would give rise to a tame subextension of the relative algebraic closures). By the uniqueness stated in Lemma 2.5, the relative algebraic closures are isomorphic over $\mathbf{K}$. ∎

Assume that $\mathbf{L}$ and $\mathbf{F}$ are henselian extensions of $\mathbf{K}$ such that $\mathbf{L}_0 \equiv_{\mathbf{K}_0} \mathbf{F}_0$. By virtue of Lemma 3.9 and the uniqueness of special models ([C–K], Thm. 5.1.17), there exist elementary extensions $\mathbf{L}^*$ and $\mathbf{F}^*$ of $\mathbf{L}$ and $\mathbf{F}$ such that $(\mathbf{L}^*)_0$ and $(\mathbf{F}^*)_0$ are isomorphic over $\mathbf{K}_0$. Since $\mathbf{L}^* \equiv_{\mathbf{K}} \mathbf{F}^*$ implies $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$, we may assume from the start that $\tau : \mathbf{L}_0 \cong_{\mathbf{K}_0} \mathbf{F}_0$. Let us take $\mathcal{T} \subset L$ and the corresponding $\mathcal{T}' \subset F$ as before; but instead of taking $\mathbf{L}'$ to be the relative algebraic closure of $(K(\mathcal{T}), v)$ in $\mathbf{L}$, we let $\mathbf{L}'$ be the maximal tame algebraic extension of the henselization $(K(\mathcal{T}), v)^h$ in $\mathbf{L}$ (which is the intersection of its absolute ramification field with $\mathbf{L}$). Then by Lemma 3.7, there is an embedding ι of $\mathbf{L}'$ into $\mathbf{F}$ over $\mathbf{K}$ which induces τ. Let $\mathbf{F}'$ be the isomorphic image of $\mathbf{L}'$ in $\mathbf{F}$. By our choice of $\mathbf{L}'$ we have that vL/vL' is a p–group and that $\overline{L}|\overline{L'}$ is purely inseparable algebraic. Since ι induces τ, it follows that the same holds for vF/vF' resp. $\overline{F}|\overline{F'}$. Thus, also $\mathbf{F}'$ admits no tame algebraic extension inside of $\mathbf{F}$. We may now replace $\mathbf{L}'$ and $\mathbf{F}'$ by their relative algebraic closures in $\mathbf{L}$ resp. $\mathbf{F}$ which are still isomorphic over $\mathbf{K}$, according to Lemma 3.12. Then the extensions $\mathbf{L}|\mathbf{L}'$ and $\mathbf{F}|\mathbf{F}'$ will be immediate since the henselian defectless Kaplansky–fields have the (RAC) property. We have thus shown:

LEMMA 3.13: *If* $\mathcal{K}$ *is an elementary class of henselian defectless Kaplansky–fields, then Lemma 3.10 and Theorem 3.11 hold without the hypothesis that* $\mathbf{K}$ *be a defectless field and* $\mathbf{L}|\mathbf{K}$ *and* $\mathbf{F}|\mathbf{K}$ *be pretame extensions.*

4. Amc–structures of level Δ, and the proof of the main theorems

Throughout this section, let $\mathbf{K}$ be a common valued subfield of $\mathbf{L}$ and $\mathbf{F}$ and Δ be a convex subgroup of vK. In the sequel, we will consider some properties of amc–structures of arbitrary level δ. Note that 4.1, 4.2, 4.3 below hold equally well for the reducts $\mathcal{O}^\delta$, G^δ and $(G^\delta, \mathrm{Pos}_\delta)$.

LEMMA 4.1: *Let γ, δ be two initial segments of vK such that $\emptyset \neq \gamma \subset \delta$. Then every isomorphism from $\mathbf{L}_\delta$ onto $\mathbf{F}_\delta$ over $\mathbf{K}_\delta$ induces an isomorphism from $\mathbf{L}_\gamma$ onto $\mathbf{F}_\gamma$ over $\mathbf{K}_\gamma$.*

Proof: Consider the ideal $\mathcal{M}^\gamma/\mathcal{M}^\delta$ of the ring $\mathcal{O}^\delta$. For γ an initial segment of vK and $\mathbf{L}$ any valued field extension of $\mathbf{K}$, we have

$$x \in \mathcal{M}_{\mathbf{L}}^\gamma/\mathcal{M}_{\mathbf{L}}^\delta \iff \forall y \in \mathcal{O}_{\mathbf{K}}^\delta \setminus (\mathcal{M}_{\mathbf{K}}^\gamma/\mathcal{M}_{\mathbf{K}}^\delta)\, \exists z \in \mathcal{M}_{\mathbf{L}}/\mathcal{M}_{\mathbf{L}}^\delta : x = yz\,.$$

Since $\mathcal{O}_{\mathbf{L}}^\delta$ is a local ring with $\mathcal{M}_{\mathbf{L}}/\mathcal{M}_{\mathbf{L}}^\delta$ as its maximal ideal, an isomorphism from $\mathbf{L}_\delta$ onto $\mathbf{F}_\delta$ over $\mathbf{K}_\delta$ carries $\mathcal{M}_{\mathbf{L}}/\mathcal{M}_{\mathbf{L}}^\delta$ onto $\mathcal{M}_{\mathbf{F}}/\mathcal{M}_{\mathbf{F}}^\delta$. Consequently, such an isomorphism will also carry $\mathcal{M}_{\mathbf{L}}^\gamma/\mathcal{M}_{\mathbf{L}}^\delta$ onto $\mathcal{M}_{\mathbf{F}}^\gamma/\mathcal{M}_{\mathbf{F}}^\delta$. Thus, also the subgroup $(1+\mathcal{M}_{\mathbf{L}}^\gamma)/(1+\mathcal{M}_{\mathbf{L}}^\delta) = \vartheta_\delta((1+\mathcal{M}_{\mathbf{L}}^\gamma)/\mathcal{M}_{\mathbf{L}}^\delta)$ of $G_{\mathbf{L}}^\delta$ is sent onto $(1+\mathcal{M}_{\mathbf{F}}^\gamma)/(1+\mathcal{M}_{\mathbf{F}}^\delta) = \vartheta_\delta((1+\mathcal{M}_{\mathbf{F}}^\gamma)/\mathcal{M}_{\mathbf{F}}^\delta)$. Since the projection of an amc–structure of level δ onto an amc–structure of level γ is obtained by reducing the ring $\mathcal{O}^\delta$ modulo the ideal $\mathcal{M}^\gamma/\mathcal{M}^\delta$ and reducing the group G^δ modulo the subgroup $(1+\mathcal{M}^\gamma)/(1+\mathcal{M}^\delta)$, it follows that every isomorphism from $\mathbf{L}_\delta$ onto $\mathbf{F}_\delta$ over $\mathbf{K}_\delta$ induces an isomorphism from $\mathbf{L}_\gamma$ onto $\mathbf{F}_\gamma$ over $\mathbf{K}_\gamma$. ∎

We will now consider the following property which is condition (iii) of Theorem 2.1 (also cited in Theorem 2.2):

$$\mathbf{L}_\delta \equiv_{\mathbf{K}_\delta} \mathbf{F}_\delta \text{ for every } \delta \in \Delta. \tag{9}$$

We want to "pull it back" to the elementary equivalence of the respective amc–structures of level Δ for elementary extensions of $\mathbf{L}$ and $\mathbf{F}$ with suitable saturation (which is condition (ii) of Theorem 2.1). From Lemma 3.9 and the uniqueness of special models (cf. [C–K], Thm. 5.1.17) we get

LEMMA 4.2: *If (9) holds, then there are elementary extensions of $\mathbf{L}$ and $\mathbf{F}$ whose amc–structures of level δ are isomorphic over $\mathbf{K}_\delta$ for every $\delta \in \Delta$.*

On the basis of this lemma, we prove

LEMMA 4.3: *If (9) holds, then there are elementary extensions of* $\mathbf{L}$ *and* $\mathbf{F}$ *whose amc–structures of level* Δ *are isomorphic over* $\mathbf{K}_\Delta$.

Proof: By virtue of the preceding lemma, we may assume w.l.o.g. that $\mathbf{L}_\delta$ is isomorphic to $\mathbf{F}_\delta$ over $\mathbf{K}_\delta$ for every $\delta \in \Delta$. We will form suitable ultrapowers of $\mathbf{L}$ and $\mathbf{F}$, using Δ as an index set. Since the collection of all final segments of Δ is closed under finite intersections, there exists an ultrafilter $\mathcal{D}$ on Δ containing all final segments (hence it is nonprincipal).

By our hypothesis on $\mathbf{L}$ and $\mathbf{F}$, we get that the ultraproducts $\prod_{\delta\in\Delta}\mathbf{L}_\delta/\mathcal{D}$ and $\prod_{\delta\in\Delta}\mathbf{F}_\delta/\mathcal{D}$ are isomorphic over $\prod_{\delta\in\Delta}\mathbf{K}_\delta/\mathcal{D}$. On the other hand, taking $\delta^* = \prod_{\delta\in\Delta}\delta/\mathcal{D}$ which is an initial segment representing an element of the value group of $\prod_{\delta\in\Delta}\mathbf{K}_\delta/\mathcal{D}$, we see that these ultraproducts are just the amc–structures of level δ^* of the elementary extensions $\mathbf{L}^* := \prod_{\delta\in\Delta}\mathbf{L}/\mathcal{D}$, $\mathbf{F}^* := \prod_{\delta\in\Delta}\mathbf{F}/\mathcal{D}$ and $\mathbf{K}^* := \prod_{\delta\in\Delta}\mathbf{K}/\mathcal{D}$ of $\mathbf{L}$, $\mathbf{F}$ and $\mathbf{K}$ respectively.

Now δ^* contains every $\delta \in \Delta$, hence $\Delta \subset \delta^*$. So by virtue of Lemma 4.1, the isomorphism of the amc–structures of level δ^* induces an isomorphism of the amc–structures of level Δ of $\mathbf{L}^*$ and $\mathbf{F}^*$ over that of $\mathbf{K}^*$ and hence also over $\mathbf{K}_\Delta$.
∎

This lemma proves that condition (iii) of Theorem 2.1 implies condition (ii). To show that condition (ii) implies condition (i), we carry on as follows. As $\mathbf{L}^*$ is an elementary extension of $\mathbf{L}$, we know that vL is pure in vL^* and $\overline{L^*}|\overline{L}$ is regular. Hence $\mathbf{L}^*|\mathbf{K}$ is a pretame extension like $\mathbf{L}|\mathbf{K}$. The same holds for $\mathbf{F}^*$ in the place of $\mathbf{L}^*$. Since $\mathbf{L}^* \equiv_{\mathbf{K}} \mathbf{F}^*$ implies $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$, we may assume from the start that $\mathbf{L}_\Delta \equiv_{\mathbf{K}_\Delta} \mathbf{F}_\Delta$. Now we want to apply Theorem 3.11 (resp. Lemma 3.13 for the case of Theorem 2.2), so we have to "switch" from the amc–structures of level Δ to the amc–structures of level 0 associated to the coarsening v_Δ of v.

LEMMA 4.4: *If* $\mathbf{L}_\Delta \equiv_{\mathbf{K}_\Delta} \mathbf{F}_\Delta$, *then*

$$(Lv_\Delta, \overline{v}_\Delta) \equiv_{(Kv_\Delta,\overline{v}_\Delta)} (Fv_\Delta, \overline{v}_\Delta) \ \textit{ and } \ (L, v_\Delta)_0 \equiv_{(K,v_\Delta)_0} (F, v_\Delta)_0 \,.$$

Proof: From $\mathbf{L}_\Delta \equiv_{\mathbf{K}_\Delta} \mathbf{F}_\Delta$ we obtain $\mathcal{O}_{\mathbf{L}}^\Delta \equiv_{\mathcal{O}_{\mathbf{K}}^\Delta} \mathcal{O}_{\mathbf{F}}^\Delta$. But these rings are just the valuation rings of $(Lv_\Delta, \overline{v}_\Delta)$, $(Fv_\Delta, \overline{v}_\Delta)$ and $(Kv_\Delta, \overline{v}_\Delta)$ respectively. The equivalence of the valuation rings implies the equivalence of the associated valued fields; this proves the first assertion.

Observe that

$$\mathcal{M}_{\mathbf{K}}^\Delta = \{a \in K \mid va > \Delta\} = \{a \in K \mid v_\Delta a > 0\} = \mathcal{M}_{(K,v_\Delta)} \,.$$

Consequently, $G_{\mathbf{K}}^{\Delta} = G_{(K,v_\Delta)}$. The same holds for $\mathbf{L}$ and $\mathbf{F}$ in the place of $\mathbf{K}$. Thus, we have

$$(10)\ (\mathcal{O}_{(Lv_\Delta,\overline{v}_\Delta)}, G_{(L,v_\Delta)}, \Theta_\Delta) \equiv_{(\mathcal{O}_{(Kv_\Delta,\overline{v}_\Delta)}, G_{(K,v_\Delta)}, \Theta_\Delta)} (\mathcal{O}_{(Fv_\Delta,\overline{v}_\Delta)}, G_{(F,v_\Delta)}, \Theta_\Delta).$$

But Kv_Δ is the fraction field of the valuation ring $\mathcal{O}_{(Kv_\Delta,\overline{v}_\Delta)}$, and Θ_Δ is just the restriction of the Θ_0–relation associated to (K, v_Δ), which we will call Θ_0'. For every $a \neq 0$ in this fraction field, a or a^{-1} is contained in the valuation ring and moreover, $\Theta_0'(a,b) \Leftrightarrow \Theta_0'(a^{-1},b^{-1})$. Again, all this holds as well for $\mathbf{L}$ and $\mathbf{F}$ in the place of $\mathbf{K}$. Hence, (10) implies our second assertion. ∎

If one of the two fields $(L, v_\Delta), (F, v_\Delta)$ is trivially valued, then it follows from the first assertion of the preceding lemma that both and also (K, v_Δ) are trivially valued and that $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$. In this case, we are done.

Now let both (L, v_Δ) and (F, v_Δ) be nontrivially valued. By condition (a) of Theorem 2.1 it then follows that $(L, v_\Delta), (F, v_\Delta) \in \mathcal{K}$. By virtue of Theorem 3.11 (resp. Lemma 3.13), the preceding lemma shows that condition (ii) of Theorem 2.1 implies

$$(11) \qquad (L, v_\Delta) \equiv_{(K,v_\Delta)} (F, v_\Delta) \ \text{ and } \ (Lv_\Delta, \overline{v}_\Delta) \equiv_{(Kv_\Delta,\overline{v}_\Delta)} (Fv_\Delta, \overline{v}_\Delta)\,.$$

So to complete the proof that condition (ii) implies condition (i), we just need the following lemma.

LEMMA 4.5: *Let $\mathbf{L}|\mathbf{K}$ and $\mathbf{F}|\mathbf{K}$ be any valued field extensions. If (11) holds, then $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$.*

Proof: By the first assertion of (11) we have $L \equiv_K F$ as fields. Let the valuation v be given by a predicate $\mathcal{O}$ for the valuation ring. Then $\mathcal{O}(x)$ holds if and only if

$$v_\Delta(x) > 0 \ \vee\ (v_\Delta(x) = 0 \ \wedge\ \overline{v}_\Delta(x/v_\Delta) \geq 0)\,.$$

So (11) implies that the equivalence of the fields holds even with the predicate for the valuation, i.e. $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$.

We leave it to the reader to adapt this proof for the case of a binary predicate for valuation divisibility (which is used in case the language does not contain a function for the multiplicative inverse). ∎

Note that this lemma does not require that the convex subgroups of vL and vF be the convex hulls of their restriction Δ to vK.

As we have already remarked in the beginning, the implication

$$\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F} \Rightarrow \forall \delta \in \Delta\colon \mathbf{L}_\delta \equiv_{\mathbf{K}_\delta} \mathbf{F}_\delta$$

follows from the fact that for every $\delta \in vK$ the formulas of the amc–structures of level δ can be encoded by formulas of the valued fields, using one constant of value δ. Hence, Theorem 2.1 and 2.2 are now completely proved, and a straightforward adaptation of the proof verifies Theorem 2.4.

Only Theorem 2.8 awaits a special treatment. The modifications which are necessary for its proof will be outlined in the sequel.

By our hypothesis that $(Kv_\Delta, \overline{v}_\Delta)$ be dense in both $(Lv_\Delta, \overline{v}_\Delta)$ and $(Fv_\Delta, \overline{v}_\Delta)$, we know that

$$\mathcal{O}^\delta_{\mathbf{L}} = \mathcal{O}^\delta_{\mathbf{K}} = \mathcal{O}^\delta_{\mathbf{F}} \text{ for every } \delta \in \Delta.$$

Let us assume in addition that

$$(G^\delta_{\mathbf{L}}, \mathrm{Pos}_\delta) \equiv_{(G^\delta_{\mathbf{K}}, \mathrm{Pos}_\delta)} (G^\delta_{\mathbf{F}}, \mathrm{Pos}_\delta) \text{ for every } \delta \in \Delta$$

which is statement (iii) of Theorem 2.8. By the procedure described at the beginning of this section, we obtain an elementary extension $(\mathbf{L}^*, \mathbf{K}^*, \mathbf{F}^*)$ of $(\mathbf{L}, \mathbf{K}, \mathbf{F})$ (since we took ultrapowers over one fixed ultrafilter), such that

$$(12)\qquad \mathcal{O}^\Delta_{\mathbf{L}^*} = \mathcal{O}^\Delta_{\mathbf{K}^*} = \mathcal{O}^\Delta_{\mathbf{F}^*} \text{ and } (G^\Delta_{\mathbf{L}^*}, \mathrm{Pos}_\Delta) \equiv_{(G^\Delta_{\mathbf{K}^*}, \mathrm{Pos}_\Delta)} (G^\Delta_{\mathbf{F}^*}, \mathrm{Pos}_\Delta)\,.$$

The first assertion shows that $(L^*v_\Delta, \overline{v}_\Delta) = (K^*v_\Delta, \overline{v}_\Delta) = (F^*v_\Delta, \overline{v}_\Delta)$. Hence, we have proved that statement (iii) of Theorem 2.8 implies statement (ii). If we are able to show that $(L^*, v_\Delta) \equiv_{(K^*, v_\Delta)} (F^*, v_\Delta)$, then by Lemma 4.5 we will obtain $\mathbf{L}^* \equiv_{\mathbf{K}^*} \mathbf{F}^*$ which implies $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$, and the first part of Theorem 2.8 will be proved.

We have already shown that $G^\Delta_{\mathbf{K}} = G_{(K, v_\Delta)}$. Furthermore, Pos_Δ is just the predicate Pos on $G_{(K, v_\Delta)}$. The same holds for $\mathbf{L}^*$, $\mathbf{K}^*$ and $\mathbf{F}^*$ in the place of $\mathbf{K}$. Now the second assertion of (12) reads as

$$(G_{(L^*, v_\Delta)}, \mathrm{Pos}) \equiv_{(G_{(K^*, v_\Delta)}, \mathrm{Pos})} (G_{(F^*, v_\Delta)}, \mathrm{Pos})\,,$$

and in view of $L^*v_\Delta = K^*v_\Delta = F^*v_\Delta$, Corollary 3.5 proves $(L^*, v_\Delta) \equiv_{(K^*, v_\Delta)} (F^*, v_\Delta)$, if we are able to show that the relative algebraic closures of (K^*, v_Δ) in (L^*, v_Δ) and (F^*, v_Δ) are tame extensions of some henselizations of (K^*, v_Δ).

By hypothesis, this holds for L, K, F in the place of L^*, K^*, F^*. In particular, this shows that $L'v_\Delta$ and $F'v_\Delta$ are separable extensions of Kv_Δ. On the other hand, condition (DENSE) yields that both of them lie in the completion of $(Kv_\Delta, \overline{v}_\Delta)$. It is known that henselian fields are separable–algebraically closed in their completions (cf. [P–Z], Corollary 7.6). Thus $(L'v_\Delta, \overline{v}_\Delta)$ and $(F'v_\Delta, \overline{v}_\Delta)$ are just henselizations of $(Kv_\Delta, \overline{v}_\Delta)$. This yields that also (L', v) and (F', v) are tame extensions of some henselizations of (K, v). We leave it to the reader to verify that then also the relative algebraic closures of $(K, v)^*$ in $(L, v)^*$ and $(F, v)^*$ are tame extensions of some henselizations of $(K, v)^*$. In view of Lemma 2.16 and Lemma 2.17 of [K1], the same holds for the coarsening v_Δ, so we are done.

Now we turn to the second part of Theorem 2.8. The proof of the implication (III) $\Rightarrow$ (II) $\Rightarrow$ (I) is similar to that of (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), except for the following modification. By hypothesis, $(L^*, v_\Delta), (F^*, v_\Delta) \in \mathcal{K}$ have divisible value groups. The elementary class of divisible ordered abelian groups is substructure complete, hence we have $v_\Delta L^* \equiv_{v_\Delta K^*} v_\Delta F^*$. Therefore, we may apply Corollary 3.5 for the proof of (II) $\Rightarrow$ (I).

The implication (IV) $\Rightarrow$ (I) will also readily follow from Corollary 3.5 and Lemma 4.5 if we show that assertion (IV) implies $\forall n \in \mathbb{N}$: $K^* \cap L^{*n} = K^* \cap F^{*n}$, where $\mathbf{K}^*$, $\mathbf{L}^*$ and $\mathbf{F}^*$ are the elementary extensions constructed by the above procedure. But the assertion $K \cap L^n = K \cap F^n$ is equivalent to

$$\forall x \in K\,(\exists y \in L\colon y^n = x \longleftrightarrow \exists z \in F : z^n = x)$$

which is an elementary sentence assumed to be valid in the structure $(\mathbf{L}, \mathbf{F}, \mathbf{K})$. Hence it is also valid in the elementary extension $(\mathbf{L}^*, \mathbf{K}^*, \mathbf{F}^*) = (\mathbf{L}, \mathbf{K}, \mathbf{F})^*$ of $(\mathbf{L}, \mathbf{K}, \mathbf{F})$. This concludes the proof of Theorem 2.8.

5. Appendix: relative completeness and model completeness

In this appendix we will use the methods of the preceding section to prove two versions of Theorem 2.1 which replace "$\equiv_{\mathbf{K}}$" by "$\prec$" and by "$\equiv$". These versions generalize theorems proved by van den Dries [VDD1]. However, it should be said that the case of henselian fields of mixed characteristic treated in [VDD1] appears indeed to be the most important application.

THEOREM 5.1: *Let $\mathcal{K}$ be an elementary class of valued fields which is model complete relative to value groups and residue fields. Let $\mathbf{L}|\mathbf{F}$ be an extension of valued fields. Suppose that Δ is a convex subgroup of vL such that*

$(L^*, v_\Delta), (F^*, v_\Delta) \in \mathcal{K}$ for all elementary extensions $\mathbf{L}^*$ and $\mathbf{F}^*$ of $\mathbf{L}$ and $\mathbf{F}$ on which v_Δ is nontrivial. Then the following statements are equivalent:

(i) $\mathbf{L} \prec \mathbf{F}$,

(ii) $(L^* v_\Delta, \overline{v}_\Delta) \prec (F^* v_\Delta, \overline{v}_\Delta)$ *and* $v_\Delta L^* \prec v_\Delta F^*$

for some elementary extension $(\mathbf{L}^*, \mathbf{F}^*) = (\mathbf{L}, \mathbf{F})^*$ *of* $(\mathbf{L}, \mathbf{F})$,

(iii) $\mathcal{O}_{\mathbf{L}}^\delta \prec \mathcal{O}_{\mathbf{F}}^\delta$ *and* $vL \prec vF$ *for every* $\delta \in \Delta$.

Proof: $\mathbf{L} \prec \mathbf{F}$ is the same as $\mathbf{L} \equiv_{\mathbf{L}} \mathbf{F}$, and $\mathcal{O}_{\mathbf{L}}^\delta \prec \mathcal{O}_{\mathbf{F}}^\delta$ is the same as $\mathcal{O}_{\mathbf{L}}^\Delta \equiv_{\mathcal{O}_{\mathbf{L}}^\Delta} \mathcal{O}_{\mathbf{F}}^\Delta$. Hence, the proof of (i) $\Rightarrow$ (iii) is straightforward and left to the reader.

If (L^*, v_Δ) is nontrivially valued, then (ii) $\Rightarrow$ (i) follows from our hypotheses that $(L^*, v_\Delta), (F^*, v_\Delta) \in \mathcal{K}$ and that $\mathcal{K}$ is model complete relative to value groups and residue fields, together with an application of Lemma 4.5. If (L^*, v_Δ) is trivially valued, then so is (F^*, v_Δ) by virtue of $v_\Delta L^* \prec v_\Delta F^*$. Then $(L^* v_\Delta, \overline{v}_\Delta) \prec (F^* v_\Delta, \overline{v}_\Delta)$ is the same as $\mathbf{L}^* \prec \mathbf{F}^*$, which in turn implies $\mathbf{L} \prec \mathbf{F}$.

It remains to prove (iii) $\Rightarrow$ (ii). As in the preceding section, $\mathcal{O}_{\mathbf{L}}^\Delta \equiv_{\mathcal{O}_{\mathbf{L}}^\Delta} \mathcal{O}_{\mathbf{F}}^\Delta$ is shown to imply

$$(L^* v_\Delta, \overline{v}_\Delta) \equiv_{(L^* v_\Delta, \overline{v}_\Delta)} (F^* v_\Delta, \overline{v}_\Delta)$$

which is just the first assertion of (ii). Furthermore, for suitably saturated extensions $\mathbf{L}^* \subset \mathbf{F}^*$ of $\mathbf{L}$ and $\mathbf{F}$, also vL^* and vF^* are highly enough saturated extensions of vL and vF. Then $vL \prec vF$ will imply that $vL^* \cong vF^*$ over vL. But such an isomorphism maps the convex hulls of Δ in vL^* and vF^* onto each other and thus induces an isomorphism $v_\Delta L^* \cong v_\Delta F^*$ over $v_\Delta L$. ∎

THEOREM 5.2: *Let $\mathcal{K}$ be an elementary class of valued fields which is complete relative to value groups and residue fields. Let $\mathbf{L}$ and $\mathbf{F}$ be two valued fields. Suppose that Δ is a common convex subgroup of vL and vF such that $(L^*, v_\Delta), (F^*, v_\Delta) \in \mathcal{K}$ for all elementary extensions $\mathbf{L}^*$ and $\mathbf{F}^*$ of $\mathbf{L}$ and $\mathbf{F}$ on which v_Δ is nontrivial. Enlarge the language of valued fields by constant symbols c_i, $i \in I$, which are interpreted in $\mathbf{L}$ and $\mathbf{F}$ such that Δ is precisely the smallest convex subgroup of vL resp. vF which contains all values vc_i. Then the following statements are equivalent:*

(i) $(\mathbf{L}, c_i)_{i\in I} \equiv (\mathbf{F}, c_i)_{i\in I}$,

(ii) $(L^* v_\Delta, \overline{v}_\Delta, c_i/v_\Delta)_{i\in I} \equiv (F^* v_\Delta, \overline{v}_\Delta, c_i/v_\Delta)_{i\in I}$ *and* $v_\Delta L^* \equiv v_\Delta F^*$

for some elementary extension $(\mathbf{L}^*, \mathbf{F}^*) = (\mathbf{L}, \mathbf{F})^*$ *of* $(\mathbf{L}, \mathbf{F})$,

(iii) $\forall \delta \in \Delta : (\mathcal{O}_{\mathbf{L}}^\delta, \pi_\delta c_i)_{i\in I} \equiv (\mathcal{O}_{\mathbf{F}}^\delta, \pi_\delta c_i)_{i\in I}$ *and* $(vL, vc_i)_{i\in I} \equiv (vF, vc_i)_{i\in I}$.

Proof: Almost everything is similar to the proof of the preceding theorem (note that Lemma 4.5 also works with "$\equiv$" in the place of "$\equiv_{\mathbf{K}}$"). Also the new constants may be handled straightforwardly. The only implication that should be pointed out is

$$(vL, vc_i)_{i\in I} \equiv (vF, vc_i)_{i\in I} \implies v_\Delta L^* \equiv v_\Delta F^* \tag{13}$$

since this is precisely why the constants were introduced. Indeed, for a suitably saturated extension $(\mathbf{L},\mathbf{F})^*$ of $(\mathbf{L},\mathbf{F})$, the left hand side of (13) implies $(vL^*, vc_i)_{i\in I} \cong (vF^*, vc_i)_{i\in I}$. This isomorphism carries the smallest convex subgroup of vL^* containing all values vc_i onto the smallest convex subgroup of vF^* containing all values vc_i. But both subgroups are just Δ, and thus, this isomorphism implies $v_\Delta L^* \cong v_\Delta F^*$. ∎

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